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# On the $\zeta$ distribution and some formula of Bessel functions (Applications of Complex Function Theory to Differential Equations)

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# On the $t$ distribution and some formula of Bessel functions

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## 1 INTRODUCTION

It is well-known that a Cauchy distribution is infinitely divisible. This is clear from the following equality:

$$\int_{-\infty}^{\infty} e^{itx} \frac{1}{\pi(1+x^2)} dx = e^{-|t|} = (e^{-|t|/n})^n = \left( \int_{-\infty}^{\infty} e^{ity} \frac{n}{\pi(1+n^2y^2)} dy \right)^n$$

for every positive integer  $n$ . That is, a Cauchy distribution can be expressed as the  $n$ -fold convolution of a Cauchy distribution with itself. In this paper we consider a probability distribution whose density is the normed product of Cauchy densities such as a following form,

$$f(a_1, \dots, a_n; x) = \frac{c}{\prod_{j=1}^n (x^2 + a_j^2)}, \quad -\infty < x < \infty$$

where  $0 < a_1 < a_2 < \dots < a_n$  and  $c$  is a normalized constant. As an interesting example we can raise a formula of Gamma function,  $x^2 |\Gamma(ix)|^2 = 1/\prod_{n=1}^{\infty} (1 + x^2/n^2)$ . The density function  $f(a_1, \dots, a_n; x)$  is an approximation of the above right hand side in the sense of weak limit, and we can look on the Student  $t$  distribution with degree of freedom  $2n - 1$  as the degenerate case of the above density function since it holds that  $f(a_1, \dots, a_n; x) \rightarrow c/(1 + x^2)^n$  as  $(a_1, \dots, a_n) \rightarrow (1, 1, \dots, 1)$ . In Section 2 we show that a distribution with density  $f(a_1, a_2; x)$  is infinite divisible. If we let  $a_1$  and  $a_2$  go to 1, we get the Student  $t$

distribution of degree of freedom 3 and from the Lévy measure of the probability distribution with the density  $f(a_1, a_2; x)$ , we can obtain the Lévy measure of the Student  $t$  distribution of degree of freedom 3 with no use of the Bessel functions. In Section 3 we will show the infinite divisibility of a probability distribution whose density is the normed product of three Cauchy densities  $f(a_1, a_2, a_3; x)$ . In this case we can obtain the Lévy measure of the Student  $t$  distribution of degree of freedom 5 with no use of the Bessel functions but general case is unsolved. The Lévy measure of the Student  $t$  distribution with any degree of freedom is obtained by E. Grosswald [4] using the Bessel functions. Works related to this paper are [2], [3], [4], [5], [6], [8]. The author was motivated by Bondesson's paper [2], which obtained original analytic results of the infinite divisibility which are connected to Steutel's integral equation and also inspired by Prof. H. M. Srivastava's lecture.

## 2 A PROBABILITY DISTRIBUTION WHOSE DENSITY IS THE NORMED PRODUCT OF TWO CAUCHY DENSITIES

Let us set  $a_1 = b, a_2 = a$  in  $f(a_1, a_2; x)$ . From  $a > b > 0$ , we see that  $f(a, b; x)$  has a nice property to make calculation simple. That is,  $f(a, b; x)$  can be written as

$$\begin{aligned}
 f(a, b; x) &= \frac{c}{a^2 - b^2} \left\{ \frac{1}{x^2 + b^2} - \frac{1}{x^2 + a^2} \right\} \\
 &= \frac{c}{a^2 - b^2} \left( \int_0^\infty e^{-(x^2 + b^2)t} dt - \int_0^\infty e^{-(x^2 + a^2)t} dt \right) \\
 &= \int_0^\infty e^{-tx^2} \frac{c}{a^2 - b^2} (e^{-b^2t} - e^{-a^2t}) dt \\
 &= \int_0^\infty \frac{1}{\sqrt{\pi v}} e^{-x^2/v} \frac{c\sqrt{\pi}}{a^2 - b^2} (e^{-b^2/v} - e^{-a^2/v}) v^{-3/2} dv. \quad (1)
 \end{aligned}$$

From the fact that the last integral is a mixture of the normal distribution, it is sufficient to show the infinite divisibility of the mixture distribution whose density is

$$g(a, b; v) = \frac{c\sqrt{\pi}}{a^2 - b^2} (e^{-b^2/v} - e^{-a^2/v}) v^{-3/2}, \quad v > 0. \quad (2)$$

The Laplace transform of  $g(a, b; v)$  is as follows:

$$\zeta(s) = \int_0^\infty e^{-sv} g(a, b; v) dv = \frac{c\pi}{a^2 - b^2} \left\{ \frac{e^{-2b\sqrt{s}}}{b} - \frac{e^{-2a\sqrt{s}}}{a} \right\}. \quad (3)$$

By analytic continuation we can extend  $\zeta(s)$  to the whole complex plane with cut along the nonpositive real line.

**Theorem 1 .** *The distribution with density  $g(a, b; v)$  is infinitely divisible. Furthermore it is a generalized Gamma convolution and the distribution with density  $f(a, b; x)$  is infinitely divisible and self-decomposable.*

*Proof.* To show the infinite divisibility of the distribution with density  $g(a, b; v)$ , it suffices to show that

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^\infty e^{-sx} k(x) dx$$

holds where  $k(x)$  is a nonnegative function. By analytic continuation of the above  $\zeta(s)$  to the whole complex plane with cut along the nonpositive real line and by the inverse Laplace transform of

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \frac{1}{e^{-2b\sqrt{s}}/b - e^{-2a\sqrt{s}}/a} \left\{ -\frac{e^{-2b\sqrt{s}}}{\sqrt{s}} + \frac{e^{-2a\sqrt{s}}}{\sqrt{s}} \right\} \\ &= \int_0^\infty e^{-sx} k(x) dx \end{aligned} \quad (4)$$

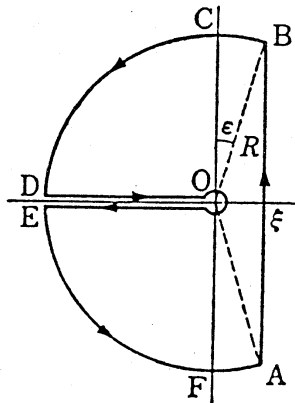
we can get the function of  $k(x)$ . For simplicity let

$$F(s) = -\zeta'(s)/\zeta(s) = \frac{1}{\sqrt{s}} \frac{1 - e^{-2(a-b)\sqrt{s}}}{\frac{1}{b} - \frac{1}{a} e^{-2(a-b)\sqrt{s}}}.$$

Suppose that (4) holds. We will calculate the inverse Laplace transform,

$$k(t) = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{\xi-iR}^{\xi+iR} e^{ts} F(s) ds, \quad \xi > 0, t > 0. \quad (5)$$

We calculate a contour integration along a curve C like the following figure:



(A) Contour integral along a small circle at O.

From

$$e^{-2(a-b)\sqrt{s}} = e^{-2(a-b)\sqrt{\rho}e^{i\theta/2}}, \quad \text{for } -\pi < \theta < \pi,$$

we see that

$$\begin{aligned} \oint e^{st} F(s) ds &= \int_{\pi}^{-\pi} e^{st} \frac{(1 - e^{-2(a-b)\sqrt{s}}) i \rho e^{i\theta}}{\sqrt{\rho} e^{i\theta/2} (\frac{1}{b} - \frac{1}{a} e^{-2(a-b)\sqrt{s}})} d\theta \\ &= \int_{\pi}^{-\pi} e^{st} \frac{(1 - e^{-2(a-b)\sqrt{s}})}{(\frac{1}{b} - \frac{1}{a} e^{-2(a-b)\sqrt{s}})} i \sqrt{\rho} e^{i\theta/2} d\theta. \end{aligned} \quad (6)$$

By  $a > b > 0$  we have

$$\left| \frac{1}{b} - \frac{1}{a} e^{-2(a-b)\sqrt{s}} \right| > \frac{1}{b} - \frac{1}{a}, \quad (-\pi \leq \theta \leq \pi),$$

and

$$\oint e^{st} F(s) ds \rightarrow 0 \text{ as } \rho \rightarrow +0.$$

(B) Integral along BD.

From  $s = Re^{i\theta}$ ,  $\sqrt{s} = \sqrt{R}(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2})$ , we see that

$$\begin{aligned} \int_{BD} e^{st} F(s) ds &= \int_{\frac{\pi}{2}-\epsilon}^{\pi} \frac{e^{st}(1 - e^{-2(a-b)\sqrt{s}})iRe^{i\theta}}{\sqrt{Re^{i\theta/2}}(\frac{1}{b} - \frac{1}{a}e^{-2(a-b)\sqrt{s}})} d\theta \\ &= i \int_{\frac{\pi}{2}}^{\pi} \sqrt{Re^{i\theta/2}} \frac{e^{Re^{i\theta}t}(1 - e^{-2(a-b)\sqrt{Re^{i\theta/2}}})}{(\frac{1}{b} - \frac{1}{a}e^{-2(a-b)\sqrt{Re^{i\theta/2}}})} d\theta \\ &+ i \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{Re^{i\theta/2}} \frac{e^{Re^{i\theta}t}(1 - e^{-2(a-b)\sqrt{Re^{i\theta/2}}})}{(\frac{1}{b} - \frac{1}{a}e^{-2(a-b)\sqrt{Re^{i\theta/2}}})} d\theta. \end{aligned} \quad (7)$$

We see that

$$\begin{aligned} \int_{\frac{\pi}{2}}^{\pi} \sqrt{R}|e^{Re^{i\theta}t}| d\theta &= \int_{\frac{\pi}{2}}^{\pi} \sqrt{R}e^{tR\cos\theta} d\theta = \int_0^{\frac{\pi}{2}} \sqrt{R}e^{-tR\sin\phi} d\phi \\ &\leq \int_0^{\frac{\pi}{2}} \sqrt{R}e^{-2tR\phi/\pi} d\phi = \sqrt{R}[-\frac{\pi}{2tR}e^{-2tR\phi/\pi}]_0^{\frac{\pi}{2}} \\ &= \sqrt{R}\{\frac{\pi}{2tR}(-e^{-tR} + 1)\} \rightarrow 0 \end{aligned} \quad (8)$$

as  $R \rightarrow +\infty$ . Next, we show that

$$\int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} |\sqrt{R}e^{i\theta/2}e^{Re^{i\theta}t}| d\theta \rightarrow 0 \quad (9)$$

as  $R \rightarrow \infty$ . From the fact that

$$\begin{aligned} \cos \theta &= \cos(\phi + \frac{\pi}{2}) = \sin(-\phi), \quad -\epsilon \leq \phi \leq 0, \\ \sin \epsilon &= \frac{\xi}{R} \geq \sin(-\phi) \geq 0, \end{aligned}$$

we see that

$$\begin{aligned} \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{R}|e^{Re^{i\theta}t}| d\theta &= \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}} \sqrt{R}e^{tR\cos\theta} d\theta \\ &\leq \int_{-\epsilon}^0 \sqrt{R}e^{tR\xi/R} d\theta = \sqrt{R}e^{t\xi}\epsilon \\ &= e^{t\xi}(\sqrt{R}\sin\epsilon)\frac{\epsilon}{\sin\epsilon} = e^{t\xi}\sqrt{R} \cdot \frac{\xi}{R} \frac{\epsilon}{\sin\epsilon} \rightarrow 0 \end{aligned} \quad (10)$$

as  $R \rightarrow \infty$ .

(C) Integrals along DO and OE.

From  $s = \rho e^{i\pi}$ ,  $\rho > 0$  on DO and  $\sqrt{s} = \sqrt{\rho} e^{i\pi/2} = i\sqrt{\rho} = iy$ ,  $\sqrt{\rho} = y$  we see that

$$\begin{aligned} \int_{DO} e^{st} F(s) ds &= \int_{DO} \frac{e^{st}}{\sqrt{s}} \left( \frac{1}{b} - \frac{1}{a} e^{-2(a-b)\sqrt{s}} \right) ds \\ &= \int_{\sqrt{R}}^0 \frac{e^{-ty^2}}{iy} \frac{1 - e^{-2(a-b)iy}}{\frac{1}{b} - \frac{1}{a} e^{-2(a-b)iy}} (-2y) dy = \frac{2}{i} \int_0^{\sqrt{R}} e^{-ty^2} \frac{1 - e^{-2(a-b)iy}}{\frac{1}{b} - \frac{1}{a} e^{-2(a-b)iy}} dy. \end{aligned} \quad (11)$$

From  $s = \rho e^{-i\pi} = -\rho$ ,  $\rho > 0$ ,  $\sqrt{s} = -i\sqrt{\rho} = -iy$ ,  $\sqrt{\rho} = y$  on OE we see that

$$\begin{aligned} \int_{OE} e^{st} F(s) ds &= \int_0^{\sqrt{R}} \frac{e^{-ty^2}}{-iy} \left( \frac{1}{b} - \frac{1}{a} e^{2(a-b)iy} \right) (-2y) dy \\ &= \frac{2}{i} \int_0^{\sqrt{R}} e^{-ty^2} \frac{1 - e^{i2(a-b)y}}{\frac{1}{b} - \frac{1}{a} e^{i2(a-b)y}} dy, \end{aligned} \quad (12)$$

and

$$\begin{aligned} &\frac{1}{2\pi i} \int_{DO+OE} \frac{e^{st}}{\sqrt{s}} \left( \frac{1}{b} - \frac{1}{a} e^{-2(a-b)\sqrt{s}} \right) ds \\ &\rightarrow -\frac{1}{\pi} \int_0^\infty e^{-ty^2} \frac{1 - e^{-i2(a-b)y}}{\frac{1}{b} - \frac{1}{a} e^{-i2(a-b)y}} dy - \frac{1}{\pi} \int_0^\infty e^{-ty^2} \frac{1 - e^{i2(a-b)y}}{\frac{1}{b} - \frac{1}{a} e^{i2(a-b)y}} dy \end{aligned} \quad (13)$$

as  $R \rightarrow \infty$ . From the Cauchy theorem we see that

$$\begin{aligned} (\text{the integral along AB})/(2\pi i) &= \frac{1}{2\pi i} \int_{\xi-iR_1}^{\xi+iR_1} e^{st} \frac{1 - e^{-2(a-b)\sqrt{s}}}{\sqrt{s}(1 - e^{-2(a-b)\sqrt{s}})} ds \\ &\rightarrow \frac{1}{\pi} \int_0^\infty e^{-ty^2} \left\{ \frac{1 - e^{i2(a-b)y}}{\frac{1}{b} - \frac{1}{a} e^{i2(a-b)y}} + \frac{1 - e^{-i2(a-b)y}}{\frac{1}{b} - \frac{1}{a} e^{-i2(a-b)y}} \right\} dy, \end{aligned} \quad (14)$$

where  $R_1 = R \cos \epsilon$ . Write the expression in the above  $\{ \}$  to a fraction. Then

$$\begin{aligned} \text{numerator} &= (1 - e^{i2(a-b)y}) \left( \frac{1}{b} - \frac{1}{a} e^{-i2(a-b)y} \right) + (1 - e^{-i2(a-b)y}) \left( \frac{1}{b} - \frac{1}{a} e^{i2(a-b)y} \right) \\ &= \frac{1}{b} - \frac{1}{b} e^{i2(a-b)y} - \frac{1}{a} e^{-i2(a-b)y} + \frac{1}{a} + \frac{1}{b} - \frac{1}{b} e^{-i2(a-b)y} - \frac{1}{a} e^{i2(a-b)y} + \frac{1}{a} \\ &= 2 \left( \frac{1}{a} + \frac{1}{b} \right) (1 - \cos 2(a-b)y), \end{aligned} \quad (15)$$

and

$$\begin{aligned} \text{denominator} &= \left( \frac{1}{b} - \frac{1}{a} e^{i2(a-b)y} \right) \left( \frac{1}{b} - \frac{1}{a} e^{-i2(a-b)y} \right) \\ &= \frac{1}{b^2} - \frac{1}{ab} e^{i2(a-b)y} - \frac{1}{ab} e^{-i2(a-b)y} + \frac{1}{a^2} \\ &= \frac{1}{a^2} + \frac{1}{b^2} - \frac{2}{ab} \cos 2(a-b)y \geq \left( \frac{1}{b} - \frac{1}{a} \right)^2. \end{aligned} \quad (16)$$

Finally we obtain

$$k(t) = \frac{1}{\pi} \int_0^\infty e^{-ty^2} \frac{2(\frac{1}{a} + \frac{1}{b})(1 - \cos 2(a-b)y)}{(\frac{1}{a} - \frac{1}{b})^2 + \frac{2}{ab}(1 - \cos 2(a-b)y)} dy, \quad t > 0. \quad (17)$$

Since the distribution with density  $g(a, b; v)$  is a generalized Gamma convolution, the mixture distribution with density  $f(a, b; x)$  is infinitely divisible and self-decomposable.  $\square$

Let

$$\pi u(y) = \frac{2(\frac{1}{a} + \frac{1}{b})(1 - \cos 2(a-b)y)}{(\frac{1}{a} - \frac{1}{b})^2 + \frac{2}{ab}(1 - \cos 2(a-b)y)}, \quad y > 0. \quad (18)$$

By the l'Hospital theorem we obtain

$$\lim_{b \rightarrow a} \pi u(y) = \lim_{b \rightarrow a} \frac{2ab(a+b)(1 - \cos 2(a-b)y)}{(a^2 + b^2) - 2ab \cos 2(a-b)y} = \frac{2^3 a^3 y^2}{1 + 2^2 a^2 y^2}, \quad (19)$$

If  $a = \frac{1}{2}$ , we have

$$u(y) = \frac{y^2}{\pi(1 + y^2)}, \quad y > 0 \quad (20)$$

and  $u(y)$  corresponds to

$$\frac{\pi^2 v}{2} M_{1+1/2}^2(v) = \frac{\pi^2 v}{2} (J_{3/2}^2(v) + Y_{3/2}^2(v)) = \frac{\pi(1 + v^2)}{v^2},$$

where  $J_{3/2}(v)$ ,  $Y_{3/2}(v)$  are the Bessel functions of the first kind and of second kind, respectively, with order  $3/2$  (cf. [1. 9.2.17]).

### 3 A PROBABILITY DISTRIBUTION WHOSE DENSITY IS THE NORMED PRODUCT OF THREE CAUCHY DENSITIES

Let us set  $a_1 = c$ ,  $a_2 = b$ ,  $a_3 = a$ ,  $c = d$  in  $f(a_1, a_2, a_3; x)$ . We will show the infinite divisibility of the distribution with density  $f(a, b, c; x)$  under the condition,  $c = b - h$ ,  $a = b + h$ ,  $h > 0$ . In the same way as the proof of the infinite divisibility of the distribution with density  $f(a, b; x)$  we have

$$f(a, b, c; x) = \int_0^\infty \frac{1}{\sqrt{\pi v}} e^{-x^2/v} g(a, b, c; v) dv,$$

where

$$g(a, b, c; v) = d\sqrt{\pi} \left\{ \frac{1}{(b^2 - c^2)(a^2 - c^2)} e^{-c^2/v} - \frac{1}{(a^2 - b^2)(b^2 - c^2)} e^{-b^2/v} + \frac{1}{(a^2 - b^2)(a^2 - c^2)} e^{-a^2/v} \right\} v^{-3/2},$$

$v > 0.$

The function  $f(a, b, c; x)$  can be expressed as the 3-fold integral,

$$f(a, b, c; x) = d \int_0^\infty e^{-tx^2} dt \int \int_{u_1 \geq 0, u_2 \geq 0, u_1 + u_2 \leq t} e^{-c^2 t - (a^2 - c^2)u_1 - (b^2 - c^2)u_2} du_1 du_2.$$

Hence  $g(a, b, c; v)$  is positive on the positive line. The Laplace transform of the mixture density  $g(a, b, c; v)$  is as follows:

$$\zeta(s) = d\pi \left[ \frac{1}{(b^2 - c^2)(a^2 - c^2)c} e^{-2c\sqrt{s}} - \frac{1}{(a^2 - b^2)(b^2 - c^2)b} e^{-2b\sqrt{s}} + \frac{1}{(a^2 - b^2)(a^2 - c^2)a} e^{-2a\sqrt{s}} \right]$$

From the above expression we obtain

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \frac{\text{numerator}}{\text{denominator}}, \\ \text{numerator} &= \frac{1}{(b^2 - c^2)(a^2 - c^2)\sqrt{s}} e^{-2c\sqrt{s}} \\ &\quad - \frac{1}{(a^2 - b^2)(b^2 - c^2)\sqrt{s}} e^{-2b\sqrt{s}} + \frac{1}{(a^2 - b^2)(a^2 - c^2)\sqrt{s}} e^{-2a\sqrt{s}}, \\ \text{denominator} &= \frac{1}{(b^2 - c^2)(a^2 - c^2)c} e^{-2c\sqrt{s}} \\ &\quad - \frac{1}{(a^2 - b^2)(b^2 - c^2)b} e^{-2b\sqrt{s}} + \frac{1}{(a^2 - b^2)(a^2 - c^2)a} e^{-2a\sqrt{s}}. \end{aligned}$$

By the inverse Laplace transform of  $-\zeta'(s)/\zeta(s)$  we can obtain the function  $k(x)$  and the following

**Theorem 2 .** *The distribution with density  $g(a, b, c; v)$  is infinitely divisible. Furthermore it is a generalized Gamma convolution and the distribution with density  $f(a, b, c; x)$  is infinitely divisible and self-decomposable.*

We can show that

$$k(t) = \int_0^\infty e^{-ty^2} u(y) dy,$$

with  $\pi u(y) = \text{numerator}/\text{denominator}$ , where

$$\begin{aligned} \text{numerator} &= (1 - \cos 2hy) \left\{ \frac{4}{(b^2 - c^2)(a^2 - b^2)b} \right. \\ &\quad + \frac{2}{(b^2 - c^2)^2} \left( \frac{1}{b} + \frac{1}{c} \right) + \frac{2}{(a^2 - b^2)^2} \left( \frac{1}{a} + \frac{1}{b} \right) \Big\} \\ &\quad + (\cos 4hy - \cos 2hy) \left( \frac{1}{a} + \frac{1}{c} \right) \frac{2}{(a^2 - b^2)(b^2 - c^2)} \geq 0 \quad (21) \end{aligned}$$



for  $y \geq 0$ ,

$$\begin{aligned} \text{denominator} &= \frac{1}{(b^2 - c^2)^2} \left( \frac{1}{c} - \frac{1}{b} \right)^2 + \frac{1}{(a^2 - b^2)^2} \left( \frac{1}{b} - \frac{1}{a} \right)^2 \\ &+ \frac{2}{(b^2 - c^2)(a^2 - b^2)b^2} + \left( \frac{2}{bc(b^2 - c^2)^2} + \frac{2}{ab(a^2 - b^2)^2} \right) (1 - \cos 2hy) \\ &- \frac{2}{b(b^2 - c^2)(a^2 - b^2)} \left( \frac{1}{c} + \frac{1}{a} \right) \cos 2hy + \frac{2 \cos 4hy}{ac(b^2 - c^2)(a^2 - b^2)} \} > 0, \quad (22) \end{aligned}$$

for  $y \geq 0$ . As we let  $h \rightarrow +0$ , we have

$$\begin{aligned} \pi u(y) &= \lim_{h \rightarrow +0} \frac{\text{numerator}}{\text{denominator}} \\ &= \frac{2^5 b^5 y^4}{3^2 + 2^2 3 b^2 y^2 + 2^4 b^4 y^4}, \quad y \geq 0 \end{aligned}$$

If  $b = \frac{1}{2}$  the above  $u(y)$  is

$$u(y) = \frac{y^4}{\pi(3^2 + 3y^2 + y^4)}, \quad y \geq 0, \quad (23)$$

which corresponds to

$$\frac{\pi^2 v}{2} M_{2+1/2}^2(v) = \frac{\pi^2 v}{2} (J_{5/2}^2(v) + Y_{5/2}^2(v)) = \frac{\pi(3^2 + 3v^2 + v^4)}{v^4}.$$

## 4 AN ANALYTIC RESULT FROM THE $t$ DISTRIBUTION

Lastly we will mention of an analytic result which comes from study of the infinite divisibility of the  $t$  distribution. Consider a  $d$ -dimensional probability distribution with density,

$$\frac{c_1}{(1 + |x|^2)^{\alpha+d/2}}, \quad x \in R^d,$$

where  $\alpha > 0$  and  $c_1$  is a normalized constant. Then we can obtain a following result.

**Theorem 3** . If  $\alpha > \frac{1}{2}$ , there exists a probability density function  $h(\alpha; x)$  such that

$$\frac{c_2}{(1 + |x|^2)^{(d+1)/2}} = \int_{R^d} \frac{c_1}{(1 + |x - y|^2)^{\alpha+d/2}} h(\alpha; y) dy,$$

holds where  $c_2$  is a normalized constant. If  $0 < \alpha < \frac{1}{2}$ , there exists a probability density function such that

$$\frac{c_1}{(1 + |x|^2)^{\alpha+d/2}} = \int_{R^d} \frac{c_2}{(1 + |x - y|^2)^{(d+1)/2}} h(\alpha; y) dy$$

holds.

Proof is omitted. The characteristic function of  $h(\alpha; x)$  is

$$\begin{aligned} \phi(t) = \exp \{ & \int_{R^d - \{0\}} (e^{tx} - 1 - \frac{itx}{1 + |x|^2}) \\ & (\frac{2}{|x|^d} \int_0^\infty \frac{1}{\pi} (1 - \frac{2}{\pi v (J_\alpha^2(v) + Y_\alpha^2(v))}) L_{d/2}(v|x|) dv) dx \} \end{aligned}$$

if  $\alpha > \frac{1}{2}$  and the integrand in the above integral corresponding to the Levy measure changes sign if  $0 < \alpha < \frac{1}{2}$ , where  $J_\alpha(v)$  and  $Y_\alpha(v)$  are Bessel functions of first and second kinds, respectively, with order  $\alpha$  and the function  $L_{d/2}(v)$  is  $L_{d/2}(v) = (2\pi)^{-d/2} v^{d/2} K_{d/2}(v)$ , where  $K_{d/2}(v)$  is the Bessel function of third kind with order  $d/2$ .

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